Math 246A Lecture 24 Notes

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November 21, 2018

1 Conformal Mappings, Uniform Convergence of Holomorphic Functions, and Normal Families

1.1 Elementary conformal mappings

Let $f : \Omega \to \Omega'$ be analytic, one to one, and onto. Then f is a conformal map from Ω to Ω'. Here are some elementary mappings

1. Let $\mathbb{D}$ be the unit disc, and let $\mathbb{H}$ be the upper half-plane. The mapping
$$Tz = \frac{z-i}{z+i}$$
is conformal from $\mathbb{H} \to \mathbb{D}$. This has inverse
$$T^{-1}z = i\frac{1-z}{1+z}.$$

2. The map $z \mapsto 1/z$ sends the interior of the unit disc to the exterior.

3. The map $z \mapsto z^{\pi/\alpha} e^{\pi/\alpha \log(z)}$ sends the complex plane to $\{z : 0 < \arg(z) < \alpha\}$.

4. Let a and b be the two points where two circles intersect. The map
$$Tz = \frac{z-a}{z-b}$$
sends the intersection of the two discs to an angle region.

5. The map
$$Tz = e^{i\alpha}\frac{z-z_0}{1-\overline{z_0}z}$$
sends the unit disc to itself and sends $z_0 \mapsto 0$.

6. Let $\mathbb{D}$ be cut along two slits along the real axis, reaching into points α, β. If we apply $z \mapsto \sqrt{z}$, we get a upper semicircle with a vertical slit in it. If we apply $z \mapsto (1+z)(1-z)$, we get the first quadrant. If we take w^2, we get the upper half plane. Rotating by i gives us the right half plane with a slit, so we have eliminated 1 of the 2 slits.

7. The mapping $z \mapsto e^z$ sends an infinite horizontal strip of height 2π to $\mathbb{C} \setminus [0, \infty)$.

8. The Joukowski transformation $w = z + 1/z$ sends $\mathbb{D}$ to $\{|z| < 1\} \cup \{z > 1\} \cup \{\infty\}$. This sends circles concentric about 0 to ellipses.

1.2 Uniform convergence of holomorphic functions

Theorem 1.1. *Let Ω be a domain, and let $f_n : \Omega \to \mathbb{C}$ be continuous such that $f_n(z) \to f(z)$ uniformly on compact subsets of Ω.*

1. *If $f_n \in H(\Omega)$, then $f \in H(\Omega)$, and $f_n' \to f'$ uniformly on compact subsets of Ω.*

2. *If the f_n are harmonic, then f is harmonic and $\nabla f_n \to \nabla f$ uniformly on compact subsets of Ω.*

Proof. To show 1, use Morera's theorem.

To show 2, we show that f has the mean value property. The integral of f_n around a disc is the value of f_n at the center. The values of f_n at the center of the disc approach the value of f around the center of the disc, and same for the value of the integrals. $\square$

Theorem 1.2. *Let $f_n \in H(\Omega)$ is such that $f_n \to f$ uniformly on compact subsets of Ω. Suppose that $f_n(z) \neq 0$ for all n for all $z \in \Omega$. Then f is either identically 0, or $f(z) \neq 0$ for all $z \in \Omega$.*

Proof. Assume $f \neq 0$ and $f(z_0) = 0$ for $z_0 \in \Omega$. Then there exists $\delta > 0$ such that $\overline{B(z_0, \delta)} \subseteq \Omega$ and $|f| > 0$ on $0 < |z - z_0| \leq \delta$. Then

$$\frac{1}{2\pi i} \oint_{|z-z_0|=\delta} \frac{f_n'(z)}{f_n(z)} \, dz = 0.$$

and this integral converges to

$$\frac{1}{2\pi i} \oint_{|z-z_0|=\delta} \frac{f'(z)}{f(z)} \, dz \neq 0.$$

This is a contradiction. $\square$

Corollary 1.1. *Suppse that $f_n \in H(\Omega)$ and $f_n \to f$ uniformly on compact subsets of Ω. If f_n is 1 to 1 on Ω for all n, then f is 1 to 1, or f is constant.*

Proof. If f is not 1 to 1, then there are 2 points $a, b \in \Omega$ where $f(a) = f(b)$. Without loss of generality, $f(a) = f(b) = 0$. There exists a simple, closed curve $\gamma \subseteq \Omega$ containing a, b. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz \geq 2.$$

This is the limit of the integrals

$$\frac{1}{2\pi i}\int_\gamma \frac{f_n'(z)}{f_n(z)}\,dz \leq 1.$$

This is a contradiction. □

1.3 Normal families

Definition 1.1. Let K be a compact metric space, and let $\mathcal{F} \subseteq C(K, \mathbb{C})$. $\mathcal{F}$ is called **precompact** if every sequence $(f_n) \subseteq \mathcal{F}$ has a subsequence f_{n_j} converging to some $f \in C(K, \mathbb{C})$ uniformly on K.

Lemma 1.1. *A family $\mathcal{F}$ is precompact if and only if*

1. *There exists M such that $\sup_K |f(x)| \leq M$ for all $f \in \mathcal{F}$.*
2. *$\mathcal{F}$ is equicontinuous: for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for $f \in \mathcal{F}$, $d_K(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon$.*

Proof. Read this some place.[1] □

Definition 1.2. Let Ω be a domain in $\mathbb{C}^*$ and $\mathcal{F} \subseteq H(\Omega)$. Then $\mathcal{F}$ is called **normal** if for every sequence (f_n) in $\mathcal{F}$, there exists a subsequence (f_{n_j}) such that $f_{n_j} \to f$ uniformly on K for every compact $K \subseteq \Omega$.

Theorem 1.3. *Let $\mathcal{F} \subseteq H(\Omega)$. Then $\mathcal{F}$ is normal if and only if for all compact $K \subseteq \Omega$, there exists $M_K < \infty$ such that $\sup_{f \in \mathcal{F}} \sup_{z \in K} |f(z)| \leq M_K$.*

Proof. ($\Longleftarrow$): This follows from the Arzelà-Ascoli theorem.
($\Longrightarrow$): We will show this next time. □

[1] You should be able to prove this on command. It's definitely fair game for qualifying exams.